Index Notation

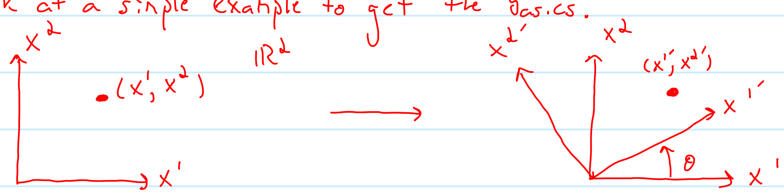
Before we go any further, we need to develop a better way to write things. So far we have made use of matrices to represent vectors, the metric and transformations. But...

3 reasons why matrices suck:

1. They are big (or can be) and writing them out explicitly can make you tired.
2. Matrices do not commute, so when we write expressions we have to be careful about order.
3. Most importantly, we will soon encounter objects and operations that cannot be represented by matrices or matrix-multiplication.

Enter... index notation. This will re-produce all of our matrix goodness and more!

Let's look at a simple example to get the basics.



In matrix terms: $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \rightarrow \begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$

Let's call $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \Lambda^{1'}{}_1 & \Lambda^{1'}{}_2 \\ \Lambda^{2'}{}_1 & \Lambda^{2'}{}_2 \end{pmatrix}$

then we can write: $x^i \rightarrow x^{i'} = \Lambda^{i'}{}_j x^j$ where we evaluate this using the Einstein Summation Convention (repeated indices are summed).

Explicitly: $x^i \rightarrow x^{i'} = \Lambda^{i'}{}_1 x^1 + \Lambda^{i'}{}_2 x^2 \Rightarrow \begin{cases} x^{1'} = \Lambda^{1'}{}_1 x^1 + \Lambda^{1'}{}_2 x^2 = \cos\theta\, x^1 + \sin\theta\, x^2 \\ x^{2'} = \Lambda^{2'}{}_1 x^1 + \Lambda^{2'}{}_2 x^2 = -\sin\theta\, x^1 + \cos\theta\, x^2 \end{cases}$

BAM!!!

Several important things to note:

1. Even in 38 dimensions, this expression would still be $x^i \to x^{i'} = \Lambda^{i'}{}_j x^j$ (size doesn't matter!)
2. We could have also written $x^i \to x^{i'} = x^j R^{i'}{}_j = x^1 R^{i'}{}_1 + x^2 R^{i'}{}_2 \Rightarrow \begin{cases} x^{1'} = x^1\cos\theta + x^2\sin\theta \\ x^{2'} = -x^1\sin\theta + x^2\cos\theta \end{cases}$

   Order doesn't matter! We get the same thing w/ $x^j R^{i'}{}_j$ or $R^{i'}{}_j x^j$. This is not the case w/ matrices!
3. You can immediately evaluate something like $M_{ijk}N^{ijk}$ given the elements of M_{ijk} and N^{ijk} even though you cannot represent this in terms of matrix multiplication (or matrices at all!)
4. You may have noticed that anytime an index is repeated it comes in an "upper" and "lower" pair, e.g. $\Lambda^{i'}{}_j x^j$. You will eventually understand what this means and why.
5. In 4D we use greek indices μ, ν, λ, etc. that take values $0, 1, 2, 3$.
   (ct, x, y, z)
6. In some cases you will see objects with one primed and one unprimed index. This is a transformation from one coordinate system to another, e.g. $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. In fact $\Lambda^{\mu'}{}_\nu$ will be the only object we encounter like this. Any other object should have all primed or all unprimed indices, e.g. $M^{\mu\nu}{}_\lambda$, $H_{\lambda'}{}^{\mu'}{}_{\nu'}$.
7. In some cases you will want to revert back to matrix expressions (when you can!). To do so we identify the row and column of a two index object as left (row) and right (column). Up or down does not matter, but to get the order right (which matters for matrices) just make sure repeated indices are immediately adjacent, e.g. $\Lambda^{\mu'}{}_\nu x^\nu$ works, but $x^\nu \Lambda^{\mu'}{}_\nu$ does not!!

   $$\begin{array}{ccc} \Lambda^{\mu'}{}_\nu x^\nu & \quad & x^\nu \Lambda^{\mu'}{}_\nu \\ \Downarrow & & \Downarrow \\ \Lambda x & & x\Lambda \end{array}$$

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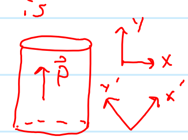
Vectors

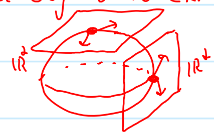
$\underbrace{\text{Spacetime vectors}}_{\text{Hitherto just "vectors"}}$ correspond to real physical quantities with magnitudes (that may or may not be related to spacetime, e.g. $\Delta\vec{x}$, $\vec{E}$) and directions that do correspond to directions in spacetime.

They have 3 important features:

1. The vector itself is invariant under coordinate changes. Sometimes this is hard to visualize so consider a momentum vector on an infinite cylinder: The vector will have different components in each coordinate system, but the actual direction does not change. If it did it could go from an ∞-direction to a periodic one!



2. The components of the vector should have a well-defined transformation under any change of coordinates used to describe the spacetime. Our main goal today is to find this law.
3. Despite that your early experience w/ vectors may have been in terms of a ray connecting two positions (e.g. a position vector w.r.t. the origin), vectors do not in general live within a space or spacetime. This is because the usual rules of vector manipulation (linear algebra) require these objects to exist in flat spaces, i.e. $\mathbb{R}^n$ or $\mathbb{M}^n$. But spaces and spacetimes can be curved, e.g. S^2



   How do we work with intrinsically flat objects in general curved spaces?
   At each point in the space we define a <u>tangent space</u> as the set of tangent vectors to all curves passing through that point. Vectors at each point in the space live in these tangent spaces.
   This has two important consequences:
   a) We cannot freely move vectors around the space since in general the tangent spaces change.
   b) Comparing vectors defined at two different points in a space will be tricky.
   Understanding how to deal w/ these two consequences will require a lot of the technology that GR rests on.

But for now, let's stick to flat space. In this case all the tangent spaces are //.

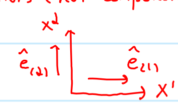
To uncover how vector components transform under coordinate changes, we first focus on a particularly simple vector $d\mathbf{s}$. The components of this vector are just coordinate differentials dx^μ, so if we know how the coordinates themselves change, then we know how the components of <u>this</u> vector change. But we do:

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu \quad \Rightarrow \quad dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\nu} dx^\nu$$

Okay, so far we know how the components of one particular vector transform. But we would like a rule for the transformation of the components of any vector. To find this we will use a trick that we will also use when we extend our discussion to general tensors, so pay attention.

Recall that a vector is invariant under coordinate transformations. It's true that components change but the whole thing should not. How do we work with this?

Recall a vector is expressed as an expansion in some basis: $d\mathbf{s} = \underbrace{dx^\mu}_{\text{components}} \underbrace{\hat{e}_{(\mu)}}_{\text{basis vectors}}$ ← labels vectors (not components)



coordinate adapted basis

The trick is that we want $d\mathbf{s}$ to be invariant, but then knowing how dx^μ transforms, we can get how $\hat{e}_{(\mu)}$ transforms.

$$d\mathbf{s} = dx^\mu \hat{e}_{(\mu)} \to d\mathbf{s}' = dx^{\mu'} \hat{e}_{(\mu')} = d\mathbf{s}$$

$$= \Lambda^{\mu'}{}_{\mu} dx^\mu \underbrace{\hat{e}_{(\mu')}}_{\text{guess } \Lambda^{\alpha}{}_{\mu'} \hat{e}_{(\alpha)}} = \Lambda^{\mu'}{}_{\mu} dx^\mu \Lambda^{\alpha}{}_{\mu'} \hat{e}_{(\alpha)} = \underbrace{\Lambda^{\alpha}{}_{\mu'} \Lambda^{\mu'}{}_{\mu} dx^\mu \hat{e}_{(\alpha)} = dx^\mu \hat{e}_{(\mu)}}$$

¿ WTF?!

$$\underbrace{\text{If } \underbrace{\Lambda^{\alpha}{}_{\mu'}}_{\Lambda^{-1}} \underbrace{\Lambda^{\mu'}{}_{\nu}}_{\Lambda} = \underbrace{\delta^{\alpha}{}_{\nu}}_{I}}$$

Remember: $\Lambda^T \eta \Lambda = \eta$

so $\Lambda^T \neq \Lambda^{-1}$!

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Don't get lost in all the index gymnastics. The end result should be straightforward. If ds is invariant, then the components and basis vectors should transform in ways that cancel each other.

In the end:

$$\boxed{\begin{aligned} &d\vec{s} = dx^{\mu}\hat{e}_{(\mu)} \rightarrow dx^{\mu'}\hat{e}_{(\mu')} = d\vec{s}' = d\vec{s} \\ &\text{if } dx^{\mu} \rightarrow dx^{\mu'} = \Lambda^{\mu'}{}_{\nu}\, dx^{\nu} \\ &\quad \hat{e}_{(\mu)} \rightarrow \hat{e}_{(\mu')} = \Lambda^{\alpha}{}_{\mu'}\, \hat{e}_{(\alpha)} \\ &\text{and } \Lambda^{\alpha}{}_{\mu'}\Lambda^{\mu'}{}_{\nu} = \delta^{\alpha}{}_{\nu} \end{aligned}}$$

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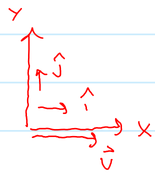
Now that we have the transformation law for the basis vectors, we recall that an arbitrary vector can be expanded as $V = V^{\mu}\hat{e}_{(\mu)}$ and should be invariant under coordinate changes. If you think about this for a few moments, hopefully you can see that this immediately tells us that the components of an arbitrary vector must transform like the components dx^{μ}! (If not, go back and review the last couple of pages carefully!)

Thus:
$$\boxed{\begin{array}{l} V = V^{\mu}\hat{e}_{(\mu)} \to V^{\mu'}\hat{e}_{(\mu')} = V' = V \\ \text{if } V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} \\ \quad \hat{e}_{(\mu')} = \Lambda^{\alpha}{}_{\mu'}\hat{e}_{(\alpha)} \\ \text{and } \Lambda^{\alpha}{}_{\mu'}\Lambda^{\mu'}{}_{\nu} = \delta^{\alpha}{}_{\nu} \end{array}}$$
Transformation law for vector components and basis vectors.

Let's see this play out in a familiar example. Consider $\longrightarrow \vec{v}$ and its description in $y\uparrow\!\!\underline{\ \ }\!\to x$ and $y' \leftarrow\!\!\underline{\ \ }\!\!\uparrow x'$.

First:



$$\vec{v} = v\hat{i} + 0\hat{j} = v^{i}\hat{e}_{(i)} \quad \text{where} \quad \hat{e}_{(1)} = \hat{i} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \hat{e}_{(2)} = \hat{j} = \begin{pmatrix}0\\1\end{pmatrix}$$
$v^1 = v,\ v^2 = 0$

Now to get to $y' \leftarrow\!\!\underline{\ \ }\!\!\uparrow x'$ we rotate the coordinates w/ $R(90) = \begin{pmatrix}\cos 90 & \sin 90\\ -\sin 90 & \cos 90\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \Rightarrow R^{i'}{}_{j}$

But to describe $\vec{v}$ in the new coordinates we first rotate the components by $R(90)$:

$$v^{i} \to v^{i'} = R^{i'}{}_{j} v^{j} \Rightarrow \begin{array}{l} v^{1} \to v^{1'} = R^{1'}{}_{1} v^{1} + R^{1'}{}_{2} v^{2} = 0 \\ v^{2} \to v^{2'} = R^{2'}{}_{1} v^{1} + R^{2'}{}_{2} v^{2} = -v \end{array}$$

But then the basis vectors must be transformed by $R(90)^{-1} = R(-90) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \Rightarrow R^{j}{}_{i'}$

$$\hat{e}_{(i)} \to \hat{e}_{(i')} = R^{j}{}_{i'}\hat{e}_{(j)} = R^{1}{}_{i'}\hat{e}_{(1)} + R^{2}{}_{i'}\hat{e}_{(2)} \Rightarrow \begin{array}{l} \hat{e}_{(1')} = R^{1}{}_{1'}\hat{e}_{(1)} + R^{2}{}_{1'}\hat{e}_{(2)} = 0\hat{e}_{(1)} + \hat{e}_{(2)} = \begin{pmatrix}0\\1\end{pmatrix} \\ \hat{e}_{(2')} = R^{1}{}_{2'}\hat{e}_{(1)} + R^{2}{}_{2'}\hat{e}_{(2)} = -\hat{e}_{(1)} + 0\hat{e}_{(2)} = \begin{pmatrix}-1\\0\end{pmatrix} \end{array}$$

Comparing we see $\hat{i}' = \hat{j}$, $\hat{j}' = -\hat{i}$

Finally: $\vec{v} = v\hat{i} + 0\hat{j} = 0\hat{i}' - v\hat{j}'$ as expected from